Adv. Appl Prob. **11**, 851–869 (1979)
Printed in N. Ireland

ON QUEUES IN DISCRETE REGENERATIVE ENVIRONMENTS, WITH APPLICATION TO THE SECOND OF TWO QUEUES IN SERIES

K. BALAGOPAL,* *Indian Statistical Institute, New Delhi*

Abstract

Let U_n be the time between the nth and $(n+1)$th arrivals to a single-server queuing system, and V_n the nth arrival's service time. There are quite a few models in which $\{U_n, V_n, n \geqq 1\}$ is a regenerative sequence. In this paper, some light and heavy traffic limit theorems are proved solely under this assumption; some of the light traffic results, and all the heavy traffic results, are new for two such models treated earlier by the author; and all the results are new for the semi-Markov queuing model.

In the last three sections, the results are applied to a single-server queue whose input is the output of a $G/G/1$ queue functioning in light traffic.

DISCRETE-PARAMETER REGENERATIVE PROCESSES; IDLE TIME; WAITING TIME; DEPARTURES; SERIES QUEUES; STRONG LIMIT AND CENTRAL LIMIT THEOREMS; LIGHT TRAFFIC AND HEAVY TRAFFIC LIMIT THEOREMS

1. Introduction

Queuing problems with non-stationary interarrival and service times have attracted much attention from analysts during the last decade and a half. Of particular theoretical interest are problems in which the variations in the arrival and service distributions are derived from some 'environmental' variations, which possess a manageable structure and take place (more or less) autonomously of the state of the system. The attractiveness of such problems stems as much from their mathematical tractability as their usefulness as models, and much work has been done on a variety of such problems. Instances, among single-server models, are the discrete Markovian environment models of Neuts [19] and others ([10], [11], [22], [1], [2], [29]); the continuous Markovian environment models of Neuts [20] and [28], [29], [21]; and what may be described as the 'additive' environment models of Grinstein and Rubinovitch [15] and Boxma [7].

As non-stationarity is a negative description, it is unlikely that such models can all be subsumed under a common theoretical head, but the search for a

Received 14 March 1978; revision received 18 January 1979.
* Postal address: Statistical Quality Control and Operations Research Unit, Indian Statistical Institute, 7 S.J.S. Sansanwal Marg, New Delhi 110 029, India.

common approach to structurally similar subclasses may be fruitful. We shall describe one such subclass here, which has a rather diverse representation, develop some results, mainly limit theorems, for it, and illustrate by applying the results to a series queuing system.

2. Description and some examples

Consider a single-counter queue which gives service on a first come first served basis. Let U_n be the interarrival time between the nth and $(n+1)$th arrivals; and V_n the nth customer's service time. Suppose $\{t_0 = 0;\, t_n,\, n \geqq 0\}$ is a discrete renewal process with strictly positive increments. We say that the system has a discrete regenerative environment if

$$\{U_n,\, V_n,\, n \geqq 1\}$$

is a regenerative process with regeneration epochs $\{t_n,\, n \geqq 0\}$. By this we mean that, for any $n \geqq 1$, the post-t_n process

$$\{U_{t_n+j},\, V_{t_n+j},\, j \geqq 1;\, t_{n+j} - t_n,\, j \geqq 0\}$$

is independent of the pre-t_n process

$$\{U_j,\, V_j,\, j \leqq t_n,\, t_j,\, j \leqq n\} \tag{2.1}$$

and all the post-t_n processes have the same law, for $n \geqq 0$. (This definition is not standard, but is convenient for our purpose; of course, we need not have *taken* $\{t_n\}$ to be a renewal process: that follows from the preceding definition.)

For some applications (e.g. the semi-Markov queue; see below) it is important to restrict the last statement in (2.1) to $n \geqq 1$; i.e. to allow the process to be 'delayed' (also 'general', cf. Smith [23]); but we shall not do so, for it makes no difference to our results, and only makes the proofs more messy.

There is a variety of apparently very different models that admit such a regenerative environment. We shall describe three such models here, and interpret the subsequent results for them. Many of our results have not hitherto been given for two of them, viz. the replacement and breakdown models, and all are new for the third, the semi-Markov model.

1. *The semi-Markov model.* This is a general version of the $M/SM/1$ model of Neuts and others, treated by Arjas [1], [2] and Takács [27]. There are a countable number of types of customers, collected into a set N. Arrivals take place in a single stream; J_n denotes the type of the $(n+1)$th arrival, and

$$\{J_0;\, U_n,\, V_n,\, J_n,\, n \geqq 1\}$$

is a Markov renewal sequence.

Fix a recurrent state $i \in N$; and let $t_n(i)$ be the epoch of nth entry of the Markov chain $\{J_n\}$ to state i. Then, from the strong Markov property, $\{U_n, V_n, n \geqq 1\}$ is regenerative, with regeneration epochs $\{t_n(i)\}$. Observe that we have a family of regeneration epochs, one for each recurrent state $i \in N$; by varying i, we may thus get information concerning customers of each type.

2. *A non-preemptive breakdown model.* This model is treated in [4]. The service mechanism exhibits 'ageing', its failure time is counted only in terms of its busy time, and failure is non-preemptive, i.e., the service during which failure occurs is completed before repair is undertaken. Let V_n now denote the so-called completion time of the nth customer, and suppose that the nth failure occurs during, or at the end of, the t_nth service. Then $\{U_n, V_n\}$ is regenerative, with regeneration epochs $\{t_n\}$ (see [4] for details).

3. *A replacement model.* This model [5] is complementary to the previous one. Here the service mechanism does not fail, but exhibits deterioration of efficiency, which is supposed to be reflected in varying service distributions. The first service has distribution B_1, the second $B_2, \ldots$ etc., until the t_1th. At the end of the t_1th service it is replaced by an identical machine, which repeats the performance. t_n is the service at the end of which the nth replacement takes place. The probability law of $\{V_n, t_n\}$ can be specified suitably so that (see [5]) $\{U_n, V_n\}$ satisfies (2.1).

3. Stability condition and some light traffic limit theorems

Let

$$X_n = V_n - U_n, \qquad n \geqq 1, \qquad S_0 = 0, \qquad S_n = S_{n-1} + X_n, \qquad n \geqq 1,$$

$$U'_n = \sum_{t_{n-1}+1}^{t_n} U_j, \qquad V'_n = \sum_{t_{n-1}+1}^{t_n} V_j, \qquad X'_n = V'_n - U'_n, \qquad n \geqq 1,$$

$$\alpha_0 = 0, \qquad \alpha_n = \alpha_{n-1} + X'_n, \qquad n \geqq 1.$$

We write E_U for $E(U'_n)$, and similarly E_V and E_X. Now, suppose W_n is the waiting time of the nth arrival. Then, as usual,

$$W_{n+1} = (W_n + X_n)^+ \tag{3.1}$$

$$= S_n - \min_{1 \leqq j \leqq n} (S_j, -W_1). \tag{3.2}$$

Busy periods, idle periods, etc. are defined as usual. In particular, idle periods are defined so that they necessarily are of positive duration. Let E_n denote the event that the nth customer ends a busy period. We say the system is stable iff $P\{E_n \text{ i.o.}\} = 1$.

We assume hereafter that $\eta = E(t_n - t_{n-1})$, E_U, E_V, are all finite and that $P(U_n' = V_n') < 1$.

Theorem 1. The system is stable iff $E_X \leqq 0$.

Proof. As is well known,

$$E_n = \left\{ S_n < \min_{1 \leqq j < n} (S_j, -W_1) \right\}. \tag{3.3}$$

Since $\{S_n\}$ is a regenerative sequence (e.g. Stidham [24]),

$$n^{-1} S_n \to \eta^{-1} E_X \quad \text{a.s.} \tag{3.4}$$

If $E_X > 0$, this implies that $S_n \to +\infty$ a.s. and hence $S_n \geqq -W_1$ for all large enough n (a.s.) so that

$$P\{E_n \text{ only for a finite } n\} = 1.$$

Suppose $E_X < 0$. Then $S_n \to -\infty$; but, on the complement of $\{E_n \text{ i.o.}\}$, $\{S_n\}$ has a (a.s.) finite minimum, hence $P\{E_n \text{ i.o.}\} = 1$. Lastly, let $E_X = 0$. Then the random walk $\{\alpha_n\}$ has zero mean, and hence (Chung and Fuchs [9]), $\liminf_{n\to\infty} \alpha_n = -\infty$, so that, *a fortiori*, $\liminf_{n\to\infty} S_n = -\infty$ a.s., which implies, as above, that a.s., E_n must happen infinitely often.

Note that $P\{E_n \text{ i.o.}\} = 0$ or 1, i.e., unlike the situation in general stationary environments (Loynes [18]) there is no quasi-stability here, though a regenerative environment is, in many ways, more complex. For instance, we do not (in general) have a limiting waiting-time distribution; but suppose $\tilde{W}_n = W_{t_n+1}$, $n \geqq 0$. For the semi-Markov model, $t_n = t_n(i)$ and since J_n denotes the type of the $(n+1)$th arrival, $\{\tilde{W}_n\}$ gives the waiting times of i-type customers. In the breakdown model, it gives the waiting times of the customers to take the first services after the successive repairs; in the replacement model, the waiting times of the customers to take the first services by the successive replacements. We shall now exhibit a limit for $\{\tilde{W}_n\}$. Let

$$\theta_n = \max_{1 \leqq j \leqq t_{n+1} - t_n} (S_{t_{n+1}} - S_{t_n+j}), \qquad n \geqq 0.$$

$\{\theta_n, n \geqq 0\}$ is an i.i.d. family of non-negative variates with $0 \leqq E(\theta_n) \leqq E_U + E_V < \infty$.

The next result is proved on the same lines as in [4], [5]. Let $\xrightarrow{\text{d}}$ denote convergence in distribution.

Theorem 2. If $E_X \geqq 0$, $\tilde{W}_n \xrightarrow{\text{d}} +\infty$ and if $E_X < 0$, $\tilde{W}_n \xrightarrow{\text{d}} \sup_{n \geqq 0} (\alpha_n + \theta_n)$. The convergence is proper.

This has some useful consequences. Index the successive arrivals as $1, 2, \ldots$. Let $\tilde{D}(t)$ denote the number of departures up to time t whose indices coincide with a regeneration epoch. For the semi-Markov model, it is (with a difference of at most unity) the number of i-type departures in time t; for the breakdown model, the number of breakdowns in time t, and for the replacement model, the number of replacements in time t.

Let $\sigma_U^2 = \operatorname{Var}(U'_n)$, and let Φ denote the standard normal distribution function.

Theorem 3. $t^{-1}\tilde{D}(t) \to \min(E_U^{-1}, E_V^{-1})$ a.s. and if $E_X < 0$, $0 < \sigma_U < \infty$, then,

$$P\{(\tilde{D}(t) - tE_U^{-1}) \leqq u\sigma_U E_U^{-3/2} t^{1/2}\} \to \Phi(u), \; -\infty < u < \infty.$$

We need one lemma. Its proof may be found in [4].

Lemma 1. $n^{-1} \min_{0 \leqq j \leqq n} S_j \to \min(0, \eta^{-1}E_X)$ a.s.

It is important that this is a purely analytical consequence of (3.4), i.e., it has nothing to do with the regenerative character of $\{U_n, V_n\}$.

Proof of Theorem 3. Let $\tilde{D}_n$ denote the time of departure of the t_nth arrival. Then

$$\tilde{D}_n = \sum_{j=1}^{t_n - 1} U_j + W_{t_n} + V_{t_n} = \sum_{j=1}^{n} U'_j + W_{t_n} + V_{t_n} - U_{t_n}. \tag{3.5}$$

Now since

$$0 \leqq V_{t_n} \leqq V'_n, \qquad 0 \leqq U_{t_n} \leqq U'_n, \tag{3.6}$$

$$n^{-1}V_{t_n}, \; n^{-1}U_{t_n} \to 0 \text{ a.s.} \tag{3.7}$$

From (3.2) and Lemma 1,

$$n^{-1}W_n \to \eta^{-1}E_X - \min(0, \eta^{-1}E_X) = \max(0, \eta^{-1}E_X) \tag{3.8}$$

so that

$$n^{-1}\tilde{D}_n \to E_U + \max(0, E_X) = \max(E_U, E_V). \tag{3.9}$$

Since $0 \leqq W_{t_n} \leqq \tilde{W}_n - X_{t_n}$, (3.6) and Theorem 2 imply that when $E_X < 0$,

$$n^{-1/2} W_{t_n} \xrightarrow{P} 0. \tag{3.10}$$

Consequently, from (3.5) and the central limit theorem for the i.i.d. sequence $\{U'_n\}$, we get

$$P\{\tilde{D}_n - nE_U \leqq u\sigma_U n^{1/2}\} \to \Phi(u). \tag{3.11}$$

Since, clearly,

$$\{\tilde{D}(t) \geqq n\} = \{\tilde{D}_n \leqq t\}, \tag{3.12}$$

(3.9) and (3.11) imply the theorem, by a technique familiar from renewal theory (see Takács [25]).

This result is new for the breakdown and replacement models. An evaluation of σ_U for those models may be found in [3].

Let $I(t)$ denote the total server idle time in $[0, t]$. The quantity limit $t^{-1}I(t)$, the average server idle time, is important; its complement under unity is the utilization factor for the system. Loynes [18] has proved its existence for systems with stationary environments; we shall show that it exists also in the presence of regenerative environments, and proceed to obtain also a central limit theorem of the type proved by Hooke [16] for the $G/G/1$ model (see also Iglehart [17]). Let

$$B_1^2 = E(V_n' - E_U^{-1}E_V U_n')^2, \qquad \sigma_V^2 = \operatorname{Var}(V_n').$$

Theorem 4. $t^{-1}I(t) \to \max(0, 1 - E_U^{-1}E_V)$ a.s. and if $E_X < 0$, σ_U, $\sigma_V < \infty$

$$P\{(I(t) - t(1 - E_U^{-1}E_V)) \leqq uB_1 E_U^{-1/2} t^{1/2}\} \to \Phi(u), -\infty < u < \infty.$$

Proof. Let

$$N(t) = \sup\left\{n \geqq 0 : \sum_{j=1}^{n} U_j \leqq t\right\} \tag{3.13}$$

so that the number of arrivals up to time t is $N(t)+1$. Let

$$s(t) = \sum_{j=1}^{N(t)} U_j.$$

Let $W(t)$ be the virtual waiting time at t. Then

$$W(t) = (W_{N(t)+1} + V_{N(t)+1} - (t - s(t)))^+. \tag{3.14}$$

From (3.8), then,

$$t^{-1}W(t) \to \eta E_U^{-1} \max(0, \eta^{-1}E_X) = E_U^{-1} \max(0, E_X) \tag{3.15}$$

and since

$$W(t) = \sum_{j=1}^{N(t)+1} V_j - t + I(t) \tag{3.16}$$

$$t^{-1}I(t) \to E_U^{-1} \max(0, E_X) + 1 - \eta^{-1}E_V \eta E_U^{-1} = \max(0, 1 - E_U^{-1}E_V) \text{ a.s.}$$

Turning to the central limit theorem, let $E_X < 0$ and consider (3.16). We need to show that $t^{-1/2}W(t) \xrightarrow{P} 0$ when σ_U, $\sigma_V < \infty$. As the proof follows rather naturally from that of a later result, we shall defer it to the end of the next section.

We may then concentrate on $\sum_{j=1}^{N(t)+1} V_j - tE_U^{-1}E_V$. Let

$$N'(t) = \sup\left\{n \geqq 0 : \sum_{j=1}^{n} U_j' \leqq t\right\}, \qquad t \geqq 0.$$

Then

$$(3.17) \quad \sum_{j=1}^{N(t)+1} V_j - tE_U^{-1}E_V = \sum_{j=1}^{N'(t)} (V_j' - E_U^{-1}E_V U_j') + \sum_{t_{N'(t)}+1}^{N(t)+1} V_j + E_U^{-1}E_V\left(\sum_{j=1}^{N'(t)} U_j' - t\right).$$

The last term has a proper limiting distribution as $t \to \infty$, and hence goes stochastically to zero when normed by $t^{1/2}$. For the second, consider that

$$t_{N'(t)+1} \geqq N(t)+1 \geqq t_{N'(t)}+1$$

so that

$$0 \leqq \sum_{t_{N'(t)}+L}^{N(t)+1} V_j \leqq V'_{N'(t)+1}$$

and it is fairly easy to show, using the finiteness of σ_V, that

$$(3.18) \qquad t^{-1/2} V'_{N'(t)+1} \xrightarrow{P} 0,$$

so that the second term may also be neglected. The summand of the first term has mean zero and finite variance B_1^2. Using the convergence of $t^{-1}N'(t)$ to E_U^{-1} and the same technique as, e.g., in Chung [8], pp. 99–100 we may show that

$$P\left\{\left(\sum_{j=1}^{N'(t)} (V_j' - E_U^{-1}E_V U_j')\right) \geqq -ut^{1/2}B_1 E_U^{-1/2}\right\} \to 1 - \Phi(-u) = \Phi(u)$$

which finishes the proof.

Finally, let $D(t)$ denote the number of departures from the system up to time t. Let

$$B_2^2 = E(U_n' - \eta^{-1}E_U(t_n - t_{n-1}))^2$$

$$\eta^{(2)} = \mathrm{Var}\,(t_n - t_{n-1}).$$

Theorem 5. $t^{-1}D(t) \to \eta \min(E_U^{-1}, E_V^{-1})$ a.s. and if $E_X < 0$, σ_U, σ_V, $\eta^{(2)} < \infty$

$$P\{(D(t) - t\eta E_U^{-1}) \leqq u\eta E_U^{-3/2} B_2 t^{1/2}\} \to \Phi(u), \qquad -\infty < u < \infty.$$

Proof. Let D_n be the time of departure of the nth customer. Then,

$$D_n = \sum_{j=1}^{n-1} U_j + W_n + V_n \tag{3.19}$$

so that, by (3.8),

$$n^{-1}D_n \to \eta^{-1}E_U + \max(0, \eta^{-1}E_X) = \eta^{-1}\max(E_U, E_V). \tag{3.20}$$

For the central limit theorem, let $E_X < 0$. It will be shown later that when σ_U, $\sigma_V < \infty$, $n^{-1/2}W_n \xrightarrow{P} 0$. Let $h(n) = \sup\{m \geqq 0 : t_m \leqq n\}$, $n \geqq 0$. Then since $0 \leqq V_n \leqq V'_{h(n)+1}$, as in (3.18), we get

$$n^{-1/2}V_n \xrightarrow{P} 0. \tag{3.21}$$

The process $\{U_n\}$ obeys the central limit theorem:

$$P\left\{\left(\sum_{j=1}^{n} U_j - n\eta^{-1}E_U\right) \leqq u\eta^{1/2}B_2 n^{1/2}\right\} \to \Phi(u)$$

(using the non-negativity of U_n, this may be deduced from a central limit theorem of Feller [13]; alternatively, a proof that does not require non-negativity can be given; see [4], Theorem 3). Hence

$$P\{(D_n - n\eta^{-1}E_U) \leqq u\eta^{-1}B_2 n^{1/2}\} \to \Phi(u). \tag{3.22}$$

Since $\{D(t) \geqq n\} = \{D_n \leqq t\}$, (3.20) and (3.22) imply the theorem, as in Theorem 3.

The central limit results of the last two theorems are new for the breakdown and replacement models. An evaluation of B_1 and B_2 may be found in [3].

It is worth mentioning that (3.18) does not really require finiteness of σ_U; that was used only to simplify the proof, since it is anyhow required elsewhere in the theorem. The same comment holds for (3.21).

4. Some heavy traffic limit theorems

First suppose $E_X > 0$, i.e., the traffic is strictly heavy. Then the total server idle time

$$\lim_{t\to\infty} I(t) < \infty \text{ a.s.} \tag{4.1}$$

This follows from the definition of stability, and Theorem 1. Alternatively,

$$\lim_{t\to\infty} I(t) = -\inf_{n\geqq 1} (S_n - W_1) < \infty \text{ a.s.}$$

Recall the balance equation (3.16):

$$W(t)=\sum_{j=1}^{N(t)+1} V_j-t+I(t).$$

In Theorem 4, we showed that $t^{-1/2}W(t)\xrightarrow{P}0$, and analysed $\sum_{j=1}^{N(t)+1} V_j-t$ to get a central limit theorem for $I(t)$. Since (4.1) easily implies that $t^{-1/2}I(t)\xrightarrow{P}0$ (a.s., in fact), we may repeat the steps therein to get the following theorem.

Theorem 6. If $E_X>0$, $0<\sigma_U$, $\sigma_V<\infty$,

$$P\{(W(t)-t(E_U^{-1}E_V-1))\leqq uB_1E_U^{-1/2}t^{1/2}\}\rightarrow\Phi(u),\qquad -\infty<u<\infty.$$

Next consider the departure times D_n, $\tilde{D}_n$. Clearly,

$$D_n=\sum_{j=1}^{n} V_j+I(D_n).$$

Using (4.1) and the central limit theorem for the regenerative sequence $\{V_n\}$ (see the observations in Theorem 5) we may prove the next result. Let

$$B_3^2=E\{V_n'-\eta^{-1}(t_n-t_{n-1})E_V\}^2.$$

Theorem 7. If $E_X>0$, $0<B_3<\infty$,

$$P\{(D(t)-t\eta E_V^{-1})\leqq uB_3\eta E_V^{-3/2}t^{1/2}\}\rightarrow\Phi(u),\qquad -\infty<u<\infty.$$

Similarly, replacing n by t_n in the preceding equation, we get the following result.

Theorem 8. If $E_X>0$, $0<\sigma_V<\infty$,

$$P\{(\tilde{D}(t)-tE_V^{-1})\leqq u\sigma_VE_V^{-3/2}t^{1/2}\}\rightarrow\Phi(u),\qquad -\infty<u<\infty.$$

Finally, we take up the 'critical' case $E_X=0$ and obtain central limit theorems for $I(t)$, $W(t)$. Note that $I(t)$, $W(t)\rightarrow+\infty$ as $t\rightarrow\infty$; we show that both diverge at rate $t^{1/2}$—a fact well known for many simpler models. Let $B_4^2=E(V_n'-U_n')^2$ (it is only B_1^2 evaluated at $E_X=0$). Let

$$\Psi(u)=(2/\pi)^{1/2}\int_0^u \exp(-y^2/2)\,dy,\qquad u\geqq 0.$$

Theorem 9. If $E_X=0$, σ_U, $\sigma_V<\infty$, then

$$P\{I(t)\leqq uB_4E_U^{-1/2}t^{1/2}\}\rightarrow\Psi(u),\qquad u\geqq 0.$$

Proof. It is easy to show that the total idle time up to the nth departure is $-\min_{1\leqq m\leqq n}(S_m,-W_1)$. Taking $W_1=0$, without loss in generality,

$$-\min_{0\leqq n\leqq D(t)-1} S_n\leqq I(t)\leqq-\min_{0\leqq n\leqq D(t)} S_n. \tag{4.2}$$

Consider

$$\min_{0\leqq n\leqq D(t)} S_n = \min_{0\leqq n\leqq D(t)} \left\{\alpha_{h(n)} + \sum_{t_{h(n)}+1}^{n} V_j - \sum_{t_{h(n)}+1}^{n} U_j\right\}$$

which implies that

$$(4.3)\qquad -\max_{0\leqq n\leqq D(t)} U'_{h(n)+1} \leqq \min_{0\leqq n\leqq D(t)} S_n - \min_{0\leqq n\leqq D(t)} \alpha_{h(n)} \leqq \max_{0\leqq n\leqq D(t)} V'_{h(n)+1}.$$

Now, $\max_{0\leqq n\leqq D(t)} U'_{h(n)+1} = \max_{1\leqq n\leqq h(D(t))+1} U'_n$ and hence (since $t^{-1}h(D(t))\to E_U^{-1}$, by Theorem 5) (4.3) implies that

$$(4.4)\qquad t^{-1/2}\left|\min_{0\leqq n\leqq D(t)} S_n - \min_{0\leqq n\leqq h(D(t))} \alpha_n\right| \xrightarrow{P} 0.$$

Since $\min_{0\leqq m\leqq n} \alpha_m$ (uncentered) obeys a central limit theorem [12], it is not very hard to show, using (4.4) and (4.2), that

$$(4.5)\qquad \liminf_{t\to\infty} P\{I(t)\leqq uB_4E_U^{-1/2}t^{1/2}\}\geqq \Psi(u), \qquad u\geqq 0.$$

Treating the other half of the inequality in (4.2) similarly, we get the theorem.

Theorem 10. If $E_X=0$, σ_U, $\sigma_V<\infty$, then

$$P\{W(t)\leqq uB_4E_U^{-1/2}t^{1/2}\}\to \Psi(u), \qquad u\geqq 0.$$

Proof. Consider (3.14):

$$W(t)=(W_{N(t)+1}+V_{N(t)+1}-(t-s(t)))^+.$$

Since $0\leqq t-s(t)\leqq t-\sum_{j=1}^{N'(t)} U'_j$, it follows that

$$(4.6)\qquad t^{-1/2}(t-s(t))\xrightarrow{P} 0.$$

Using (3.2) we may write

$$(4.7)\qquad W_{N(t)+1}+V_{N(t)+1}=\alpha_{N'(t)}-\min_{0\leqq j\leqq N(t)} S_j+\sum_{t_{N'(t)+1}}^{N(t)} X_j+V_{N(t)+1}.$$

As in (3.18), using finiteness of σ_U, σ_V, we find that

$$(4.8)\qquad t^{-1/2}\left|\sum_{t_{N'(t)}+1}^{N(t)} X_j\right|, \qquad t^{-1/2}V_{N(t)+1}\xrightarrow{P} 0.$$

Next, since $t^{-1}N(t)\to \eta E_U^{-1}$, like $t^{-1}D(t)$ in Theorem 9,

$$(4.9)\qquad t^{-1/2}\left|\min_{0\leqq n\leqq N(t)} S_n - \min_{0\leqq n\leqq h(N(t))} \alpha_n\right| \xrightarrow{P} 0.$$

But as $h(N(t))=N'(t)$, (3.14), (4.6)–(4.9) show that

$$\lim_{t\to\infty} P\{W(t)\leqq ut^{1/2}\}=\lim_{t\to\infty} P\left\{\alpha_{N'(t)}-\min_{0\leqq n\leqq N'(t)} \alpha_n\leqq ut^{1/2}\right\}. \tag{4.10}$$

We now use a variant of a familiar combinatorial trick. We have

$$P\left\{\alpha_{N'(t)}-\min_{0\leqq n\leqq N'(t)} \alpha_n\leqq u\right\}=\sum_{m=0}^{\infty} P\left\{\alpha_m-\min_{0\leqq n\leqq m} \alpha_n\leqq u;\ \sum_{j=1}^{m} U'_j\leqq t<\sum_{j=1}^{m+1} U'_j\right\}.$$

Consider $(U'_1, V'_1), (U'_2, V'_2), \cdots, (U'_m, V'_m)$. These are independent, identically distributed, pairs. Take them in the reverse order, i.e., permute this collection as $(U'_m, V'_m), (U'_{m-1}, V'_{m-1}), \cdots, (U'_1, V'_1)$. Then

$$P\left(\alpha_m-\min_{0\leqq n\leqq m} \alpha_n\leqq u;\ \sum_{j=1}^{m} U'_j\leqq t<\sum_{j=1}^{m+1} U'_j\right)$$
$$=P\left\{\max_{0\leqq n\leqq m} \alpha_n\leqq u;\ \sum_{j=1}^{m} U'_j\leqq t<\sum_{j=1}^{m+1} U'_j\right\}$$

and summing over m,

$$P\left\{\alpha_{N'(t)}-\min_{0\leqq n\leqq N'(t)} \alpha_n\leqq u\right\}=P\left\{\max_{0\leqq n\leqq N'(t)} \alpha_n\leqq u\right\}. \tag{4.11}$$

The rest of the proof is as in the previous theorem.

All the preceding results are new for the replacement and breakdown models. Evaluations of B_3, B_4 and σ_V may be found in [3].

Before ending, we turn again to light traffic theory, and fill in the gaps in the proofs of Theorems 4 and 5.

Lemma 1. If $E_X<0$, σ_U, $\sigma_V<\infty$, $t^{-1/2}W(t)\xrightarrow{P}0$, $n^{-1/2}W_n\xrightarrow{P}0$ as t, $n\to+\infty$, respectively.

Proof. The first is almost proved in the previous theorem. As there, we have, under the hypothesis, (see (4.10) and (4.11)), for any $\delta>0$

$$\lim_{t\to\infty} P\{W(t)\geqq t^{1/2}\delta\}=\lim_{t\to\infty} P\left\{\max_{0\leqq n\leqq N'(t)} \alpha_n\geqq t^{1/2}\delta\right\}=0$$

since $\max_{0\leqq n\leqq N'(t)} \alpha_n$ grows to $\sup \alpha_n$, which is finite.

To prove the second, consider (3.2). Proceeding as in the previous theorem we may show that

$$\lim_{n\to\infty} P\{W_n\geqq \delta n^{1/2}\}=\lim_{n\to\infty} P\left\{\alpha_{h(n)}-\min_{0\leqq j\leqq h(n)} \alpha_j\geqq \delta n^{1/2}\right\}$$
$$=\lim_{n\to\infty}\sum_{m=0}^{\infty} P\left\{\alpha_m-\lim_{0\leqq j\leqq n} \alpha_j\geqq \delta n^{1/2};\ t_m\leqq n<t_{m+1}\right\}.$$

Permute in reverse order the i.i.d. triples (U'_1, V'_1, t_1), $(U'_2, V'_2, t_2 - t_1), \ldots, (U'_m, V'_m, t_m - t_{m-1})$ to get

$$P\left\{\alpha_m - \min_{0 \leqq j \leqq m} \alpha_j \geqq \delta n^{1/2};\ t_m \leqq n < t_{m+1}\right\} = P\left\{\max_{0 \leqq j \leqq m} \alpha_j \geqq \delta n^{1/2};\ t_m \leqq n < t_{m+1}\right\}.$$

The rest of the proof is as above.

Remark 1. For the $G/G/1$ (and many other) models, $W(t)$ and W_n have proper limiting distributions as $t, n \to +\infty$ when $E_X < 0$. And this, of course, implies the conclusion in Lemma 1. We may therefore expect that for our model too, at least this much is true without any additional conditions. For the semi-Markov model this can be established provided that the total number of types of customers is finite (as e.g. in [19], [22]); for the replacement model, provided the number of services given by each replacement is bounded (see [3] for details). Unfortunately, it appears difficult to establish it generally. This does no harm to Theorem 4, since the auxiliary condition σ_U, $\sigma_V < \infty$ is anyhow required there, but the same cannot be said of Theorem 5. Obviously, the condition $\sigma_V < \infty$ ought to be irrelevant then (for instance, the corresponding result for the $G/G/1$ model, Iglehart [17], does not require finiteness of service-time variance).

5. The second of two queues in series: light traffic theory

Consider two queues in series; the first is $G/G/1$ and the second has the departures from the first as input, a general service distribution at a single server working to a first come first served rule, and infinite queue capacity.

Let C denote a generic 1-busy cycle duration, I, a 1-idle period duration, and N the number of services given during a 1-busy period, where the prefix 1 refers to the first queue. We use the following facts about these quantities (see e.g. Iglehart [17], Lemma 2.3). Let λ^{-1} and μ_1^{-1} denote the mean interarrival and service times to the first queue. We assume throughout that $\lambda < \mu_1$, i.e., the first queue is in light traffic.

Lemma 2.

$$E(I) = E(N)(\lambda^{-1} - \mu_1^{-1})$$
$$E(C) = E(N)\lambda^{-1}.$$

$E(N)$, $E(C)$ and $E(I)$ are all finite.

Hereafter, we shall write E_N for $E(N)$. Let G denote the service-time distribution at the second server. We define input to it in the following manner: let $N_1, N_2, \cdots$ be the number of services given in the 1-busy cycles numbered

$1, 2, \cdots$. For each $n \geqq 0$, at the time of commencement of the $(n+1)$th 1-busy cycle (at the time of entry of customer numbered $\sum_{j=1}^{n} N_j + 1$ to the first queue), we introduce a fictitious customer who proceeds directly to the second queue and queues up. His service requirement there is (identically) zero. Clearly, this does not alter either the virtual waiting time or the idle time in the second queue—only the queue length and departure processes.

Let $t_0 = 0$; $t_n = \sum_{j=1}^{n} N_j + n$, $n \geqq 1$. U_{t_n+1} is the duration from the commencement of the $(n+1)$th 1-busy cycle to the first departure from the first queue during the $(n+1)$th 1-busy cycle (it is actually the first service duration); $U_{t_n+2}, \cdots U_{t_{n+1}-1}$ give the interdeparture times from the first queue during the $(n+1)$th 1-busy cycle; and $U_{t_{n+1}}$ is the $(n+1)$th 1-idle period duration.

The customers numbered $t_n + 1$ are fictitious; so we take $V_{t_n+1} \equiv 0$; and $V_{t_n+2}, \cdots, V_{t_{n+1}}$ are obtained by sampling randomly N_{n+1} times from a population with distribution G. It is now clear that, since busy period initiation is a recurrent event for the $G/G/1$ queue, $\{U_n, V_n\}$ is a regenerative sequence, with regeneration epochs $\{t_n, n \geqq 0\}$.

Let μ_2^{-1} and $\mu_2^{(2)}$ be the mean (finite) and variance of G, $\lambda^{(2)}$ the variance of the interarrival times to the first queue. Let $\sigma^2(C)$ $\sigma^2(N)$, be the variances of C and N, and $\sigma(C, N)$ their covariance.

Lemma 3. $E_U = \lambda^{-1} E_N$, $E_V = \mu_2^{-1} E_N$.

Proof. $E_U = E\{\sum_{t_n+1}^{t_{n+1}} U_j\}$; and as explained above, $\sum_{t_n+1}^{t_{n+1}} U_j$ is nothing but the $(n+1)$th 1-busy cycle duration. By Lemma 2, then, $E_U = \lambda^{-1} E_N$. And

$$E_V = E\left\{\sum_{t_n+1}^{t_{n+1}} V_j\right\} = 0 + E(t_{n+1} - t_n - 1)\mu_2^{-1} = \mu_2^{-1} E_N.$$

Remark 2. The first two moments of C and N play an important role in the analysis. We note that when the input to the first queue is Poisson, they are easily evaluated, using the well-known functional equation of Takács [26]. We get

$$E_N = (1 - \lambda\mu_1^{-1})^{-1}$$
$$\sigma^2(N) = (\lambda^2 \mu_1^{(2)} + \lambda\mu_1^{-1})(1 - \lambda\mu_1^{-1})^{-3}$$
$$\sigma^2(C) = (\mu_1^{(2)} + \lambda\mu_1^{-3})(1 - \lambda\mu_1^{-1})^{-3} + \lambda^{-2}$$

and

$$\sigma(C, N) = \lambda(\mu_1^{(2)} + \mu_1^{-2})(1 - \lambda\mu_1^{-1})^{-3}$$

where $\mu_1^{(2)}$ is the variance of the first server's service time.

Theorem 11. The second queue is stable iff $\lambda \leqq \mu_2$.

Proof. By Lemma 3, $E_X = E_N(\mu_2^{-1} - \lambda^{-1})$, and since $E_N < \infty$, the assertion follows from Theorem 1.

Consider $\tilde{W}_n$: it gives the waiting time of the nth fictitious customer, i.e., the virtual waiting time at the second server at the start of the $(n+1)$th 1-busy cycle. It can be regarded as the part of the first n 1-busy cycles' load still to be cleared by the second server at the start of the $(n+1)$th 1-busy period.

Theorem 2 gives us the following result.

Theorem 12. If $\lambda \geqq \mu_2$, $\tilde{W}_N \xrightarrow{d} +\infty$ and if $\lambda < \mu_2$, $\tilde{W}_n$ converges in distribution to a finite-valued random variable.

The form of the limit is also given there but is not of much use.

Similarly, $\tilde{D}(t)$ gives the number of 1-busy cycle loads completely cleared by the second server up to time t.

Theorem 13. $t^{-1}\tilde{D}(t) \to E_N^{-1} \min(\lambda, \mu_2)$ a.s. and, if $\lambda < \mu_2$, $\sigma(C) < \infty$,

$$P\{(\tilde{D}(t) - t\lambda E_N^{-1}) \leqq uat^{1/2}\} \to \Phi(u), -\infty < u < \infty,$$

where $a^2 = \sigma(C)\lambda^3 E_N^{-3}$.

Proof. By virtue of Theorem 3 and Lemma 3, it is only necessary to note that $\sigma_U = \sigma(C)$.

We note that, here and in all the central limit theorems to follow, in the case of Poisson input (to the first queue), the finiteness condition on the norming term is equivalent to $\mu_1^{(2)} < \infty$.

Next consider the second server's idle time, $I(t)$. Theorem 4 and Lemma 3 above give us the following result. The evaluation of B_1 is straightforward, if tedious.

Theorem 14. $t^{-1}I(t) \to \max(0, 1 - \lambda\mu_2^{-1})$ a.s. and if $\lambda < \mu_2$, $\sigma(N)$, $\sigma(C)$ and $\mu_2^{(2)} < \infty$,

$$P\{(I(t) - t(1 - \lambda\mu_2^{-1})) \leqq ubt^{1/2}\} \to \Phi(u), \qquad -\infty < u < \infty,$$

where $b^2 = \lambda\mu_2^{(2)} + \lambda E_N^{-1}\mu_2^{-2}(\sigma^2(N) + \lambda^2\sigma^2(C)) - 2\lambda^2\mu_2^2 E_N^{-1}\sigma(N, C)$.

Lastly, consider the process of departures from the second queue (and hence the system), $\{D(t), t \geqq 0\}$. We cannot directly use Theorem 6, for the fictitious customers would then get included. We shall therefore give a different proof.

Theorem 15. $t^{-1}D(t) \to \min(\lambda, \mu_2)$ a.s. and if $\lambda < \mu_2$, $\sigma(C)$, $\sigma(N) < \infty$, then

$$P\{(D(t) - \lambda t) \leqq uct^{1/2}\} \to \Phi(u), \qquad -\infty < u < \infty,$$

where $c^2 = \lambda^{(2)}\lambda^3$.

Proof. Let D'_n denote the time of arrival to the second queue of the nth *actual* customer, $W_n^{(a)}$ his waiting time there and $V_n^{(a)}$ his service time. Then, even though $\{D'_n\}$ is *not* regenerative, we still have

$$n^{-1}D'_n \to \max(\lambda^{-1}, \mu_1^{-1}) = \lambda^{-1} \quad \text{a.s.} \tag{5.1}$$

and if $\lambda^{(2)} < \infty$, then

$$P\{(D'_n - n\lambda^{-1}) \leqq u(\lambda^{(2)})^{1/2} n^{1/2}\} \to \Phi(u), \qquad -\infty < u < \infty \tag{5.2}$$

(Equations (3.22) and (3.24), particularized to the $G/G/1$ queue). And $D_n^{(a)}$, the nth (actual) customer's departure time from the second queue, satisfies:

$$D_n^{(a)} = D'_n + W_n^{(a)} + V_n^{(a)}. \tag{5.3}$$

Now, $W_n^{(a)} = W(D'_n)$, where W, as before, refers to the virtual wait in the second queue. We capitalize on the fact that W is not affected by the introduction of fictitious customers. Consequently, (3.17) and (5.1) imply that

$$n^{-1}D_n^{(a)} \to \max(\lambda^{-1}, \mu_2^{-1})$$

which implies, as before, that $t^{-1}D(t) \to \min(\lambda, \mu_2)$.

For the central limit theorem, all we need, in view of (5.3) and (5.2) is that $n^{-1/2}W(D'_n) \xrightarrow{P} 0$. From (3.14),

$$W(D'_n) = (W_{N(D'_n)+1} + V_{N(D'_n)+1} - (D'_n - s(D'_n)))^+. \tag{5.4}$$

Now, $D'_n - s(D'_n)$ is not larger than the time between the start of the 1-busy cycle during which the nth customer gets his service (at the first server) and his departure from the first queue; it is, therefore, not larger than the total duration of the 1-busy cycle during which the nth customer gets his service, which latter quantity has a proper limiting distribution as $n \to +\infty$. Hence

$$n^{-1/2}(D'_n - s(D'_n)) \xrightarrow{P} 0. \tag{5.5}$$

The remaining two terms in (5.4) may be treated as in Theorem 10, to get the analogue of (4.10) there, with $W(D'_n)$ in place of $W(t)$, and $N'(D'_n)$ in place of $N'(t)$. So consider, for $\delta > 0$,

$$\begin{aligned} &P\left\{\alpha_{N'(D'_n)} - \min_{0 \leqq m \leqq N'(D'_n)} \alpha_m > \delta n^{1/2}\right\} \\ &= \sum_{r=0}^{\infty} P\left\{\alpha_r - \min_{0 \leqq m \leqq r} \alpha_m > \delta n^{1/2};\ \sum_{j=1}^{r} U'_j \leqq D'_n < \sum_{j=1}^{r+1} U'_j\right\} \\ &= \sum_{r=0}^{\infty} P\left\{\alpha_r - \min_{0 \leqq m \leqq r} \alpha_m > \delta n^{1/2};\ \sum_{j=1}^{r} N_j < n \leqq \sum_{j=1}^{r+1} N_j\right\} \end{aligned}$$

(to see the validity of this, observe that $\sum_{j=1}^{r} U'_j$ is the time of starting of the

$(r+1)$th 1-busy cycle, and $\sum_{j=1}^{r} N_j$ is the number of services given during the first r 1-busy cycles)

$$= \sum_{r=0}^{\infty} P\left\{\alpha_r - \min_{0 \leqq m \leqq r} \alpha_m > \delta n^{1/2};\, t_r < n + r < t_{r+1}\right\}.$$

The proof is now finished using the same interchange as in Lemma 1.

6. Heavy traffic theory

Theorems 6 and 8 are immediately reproduced.

Theorem 16. If $\lambda > \mu_2$, $\sigma(C)$, $\sigma(N)$ and $\mu_2^{(2)} < \infty$

$$P\{(W(t) - t(\lambda\mu_2^{-1} - 1)) \leqq ubt^{1/2}\} \to \Phi(u), \qquad -\infty < u < \infty,$$

where $b^2 = \lambda\mu_2^{(2)} + \lambda E_N^{-1}\mu_2^{-2}(\sigma^2(N) + \lambda^2\sigma^2(C)) - 2\lambda^2\mu_2^2 E_N^{-1}\sigma(N, C)$.

Theorem 17. If $\lambda > \mu_2$, $\sigma(N)$, $\mu_2^{(2)} < \infty$,

$$P\{(\tilde{D}(t) - tE_N^{-1}\mu_2) \leqq upt^{1/2}\} \to \Phi(u), \qquad -\infty < u < \infty,$$

where $p^2 = \mu_2 E_N^{-2}(\mu_2^2\mu_2^{(2)} + E_N^{-1}\sigma^2(N))$.

Next consider the departure process $\{D(t)\}$. We cannot directly use Theorem 7, but our job is almost as easy. We have

$$D_n^{(a)} = \sum_{j=1}^{n} V_j^{(a)} + I(D_n^{(a)})$$

and since I, like W, is invariant under the introduction of fictitious customers, (4.1) still holds, and so we get the following result.

Theorem 18. If $\lambda > \mu_2$, $\mu_2^{(2)} < \infty$,

$$P\{(D(t) - t\mu) \leqq uqt^{1/2}\} \to \Phi(u), \qquad -\infty < u < \infty,$$

where $q^2 = \mu_2^{(2)}\mu_2^3$.

In the critically heavy traffic case, we have a further result.

Theorem 19. If $\lambda = \mu_2$, $\sigma(C)$, $\sigma(N)$, $\mu_2^{(2)} < \infty$

$$P\{W(t) \leqq urt^{1/2}\} \to \Psi(u), \qquad u \geqq 0$$

$$P\{I(t) \leqq urt^{1/2}\} \to \Psi(u), \qquad u \geqq 0$$

where $r^2 = \lambda\mu_2^{(2)} + \lambda^{-1}E_N^{-1}\sigma^2(N) + \lambda\sigma^2(C) - 2E_N^{-1}\sigma(N, C)$.

Remark 3. We could complicate the model without much trouble. For instance, it may be that the customers who have zero waiting time at the first server take extra service at the second. More generally, we could introduce any sort of dependence between the 2-service times of customers served in a *single*

1-busy cycle, and still retain the formal analysis intact. Only, the evaluations of the constants B_1, B_2 etc. in terms of the 1-busy cycle quantities may no longer be possible.

7. Utilisation factor for m queues in series

Our general analysis, unfortunately, is restricted to the second of two queues in series: the output of the second queue is not regenerative. We could try to pick those points at which both a 1-busy cycle and a 2-busy cycle are initiated (the latter, of course, by a fictitious customer) but there is no guarantee that this will generate a recurrent event.

But there are two important results that used only the strong limit for the input $(t^{-1}N(t) \to \lambda)$ and not its full regenerative nature, and since this limit (in light traffic) re-produces itself at the output $(t^{-1}D(t) \to \lambda)$, these results are capable of generalization.

Consider then m queues in series. Let $D_n(r)$ be the time of the nth customer's departure from the rth queue; $D_r(t)$, the number of departures from the rth queue up to time t. Let $I_r(t)$ be the rth server's idle time up to time t. Let $\{V_n(r), n \geqq 1\}$, the service times at the rth server, form an i.i.d. family with finite mean μ_r^{-1}. We could treat all possible combinations of light and heavy traffic at each server, but shall deal only with the case of 'stability' at each server, i.e., $\lambda \leqq \mu_r$, $r = 1, 2, \cdots, m$.

Theorem 20. If $\lambda \leqq \mu_r$, $r = 1, 2, \cdots, m$, then

$$t^{-1}D_r(t) \to \lambda$$

and

$$t^{-1}I_r(t) \to 1 - \lambda\mu_r^{-1} \quad \text{a.s.}$$

for $r = 1, 2, \cdots, m$.

Proof. We know the result to be true for $r = 1$ (even $r = 2$, for that matter). We shall prove the general result by induction. Suppose it to be true for the $(r-1)$th queue. Let $W_n(r)$ denote the nth customer's waiting time at the rth server. Then

$$W_n(r) = S_{n-1}(r) - \min_{0 \leqq j \leqq n-1} S_j(r)$$

where

$$S_m(r) = \sum_{j=1}^{m} V_j(r) - D_{m+1}(r-1),\ m \geqq 1; \qquad S_0(r) \equiv 0.$$

As a consequence of the induction hypothesis,

$$n^{-1}S_{n-1}(r) \to \mu_r^{-1} - \lambda^{-1} \quad \text{a.s.}$$

and since the limit is $\leqq 0$ (cf. Lemma 1 and the comment following it),

$$n^{-1} \min_{0 \leqq j \leqq n-1} S_j(r) \twoheadrightarrow \mu_2^{-1} - \lambda^{-1}$$

and hence

$$n^{-1} W_n(r) \to 0 \quad \text{a.s.} \tag{7.1}$$

Now,

$$D_n(r) = D_{n-1}(r) + W_n(r) + V_n(r)$$

so that $n^{-1} D_n(r) \to \lambda^{-1}$, which implies that

$$t^{-1} D_r(t) \to \lambda \quad \text{a.s.}$$

Let $W_r(t)$ denote the virtual waiting time at the rth server. Clearly, $0 \leqq W_r(t) \leqq W_{D_{r-1}(t)}(r)$ so that, by (7.1) and the induction hypothesis,

$$t^{-1} W_r(t) \to 0 \quad \text{a.s.} \tag{7.2}$$

From the balance equation

$$W_r(t) = \sum_{j=1}^{D_{r-1}(t)} V_j - t + I_r(t)$$

and (7.2), we get

$$t^{-1} I_r(t) \to 1 - \lambda \mu_r^{-1} \quad \text{a.s.}$$

which completes the proof.

In other words, the rth server's utilization factor is $\lambda \mu_r^{-1}$. Hence, we have evaluated the following.

Corollary 1. The utilization factor for a series queuing system with unbounded queue capacity at each server is

$$\lambda \sum_{r=1}^{m} \mu_r^{-1}$$

provided that $\lambda \leqq \mu_r$, $r = 1, 2, \cdots, m$.

In other words, as far as utilization goes, it is as if the first server himself gives all the m services—though the stability criterion is much less liberal then. For another instance of such a 'conservation' of utilization, see [6].

References

[1] ARJAS, E. (1972) On a fundamental identity in the theory of semi-Markov processes. *Adv. Appl. Prob.* **4,** 258–270.

[2] Arjas, E. (1972) On the use of a fundamental identity in the theory of semi-Markov queues. *Adv. Appl. Prob.* **4,** 271–284.

[3] Balagopal, K. (1978) Limit Theory For Some Queuing and Storage Models With Stochastic Environments, Ph.D. thesis submitted to Kakatiya University, Warangal, India.

[4] Balagopal, K. (1977) Limit theorems for the single server queue with non-preemptive service interference. *Opsearch* **14,** 244–263.

[5] Balagopal, K. (1977) A generalised random walk in the theory of queues and dams. *Sankhyā* A**39,** 264–277.

[6] Balagopal, K. (1978) A conservation law for utilisation in queues, *Opsearch* **15,** 12–21.

[7] Boxma, O. J. (1975) The single-server queue with random service output. *J. Appl. Prob.* **12,** 763–778.

[8] Chung, K. L. (1967) *Markov Chains With Stationary Transition Probabilities.* Springer-Verlag, New York.

[9] Chung, K. L. and Fuchs, W. H. J. (1951) On the distribution of values of sums of random variables. *Mem. Amer. Math. Soc.* **6.**

[10] Çinlar, E. (1967) Time-dependence of queues with semi-Markovian service times. *J. Appl. Prob.* **4,** 356–364.

[11] Çinlar, E. (1967) Queues with semi-Markovian arrivals. *J. Appl. Prob.* **4,** 365–379.

[12] Erdös, P. and Kac, M. (1946) On certain limit theorems of the theory of probability. *Bull. Amer. Math. Soc.* **52,** 292–302.

[13] Feller, W. (1949) Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* **67,** 98–119.

[14] Feller, W. (1966) *An Introduction to Probability Theory and its Applications,* II. Wiley Eastern, New Delhi.

[15] Grinstein, J. and Rubinovitch, M. (1974) Queues with random service output: the case of Poisson arrivals. *J. Appl. Prob.* **11,** 771–784.

[16] Hooke, J. A. (1970) On some limit theorems for the *GI/G/*1 queue. *J. Appl. Prob.* **7,** 634–640.

[17] Iglehart, D. L. (1971) Functional limit theorems for the queue *GI/G/*1 in light traffic. *Adv. Appl. Prob.* **3,** 269–281.

[18] Loynes, R. M. (1962) The stability of a queue with non-independent interarrival and service times. *Proc. Camb. Phil. Soc.* **58,** 497–520.

[19] Neuts, M. F. (1966) The single-server queue with Poisson input and semi-Markovian service times. *J. Appl. Prob.* **3,** 202–230.

[20] Neuts, M. F. (1971) A queue subject to extraneous phase changes. *Adv. Appl. Prob.* **3,** 78–119.

[21] Purdue, P. (1974) The *M/M/*1 queue in a Markovian environment. *Operat. Res.* **22,** 562–569.

[22] Purdue, P. (1975) A queue with Poisson input and semi-Markov service times: busy period analysis. *J. Appl. Prob.* **12,** 353–357.

[23] Smith W. L. (1955) Regenerative stochastic processes. *Proc. R. Soc. London* A**232,** 6–31.

[24] Stidham, S. (1972) Regenerative stochastic processes in the theory of queues, with application to alternating priority queue. *Adv. Appl. Prob.* **4,** 542–577.

[25] Takács, L. (1956) On a probability problem arising in the theory of counters. *Proc. Camb. Phil. Soc.* **32,** 488–498.

[26] Takács, L. (1967) *Combinatorial Methods in the Theory of Stochastic Processes.* Wiley, New York.

[27] Takács, L. (1976) A Banach space of matrix functions and its application in the theory of queues. *Sankhyā* A **38,** 201–211.

[28] Yechiali, U. (1973) A queuing type birth and death process defined on a continuous-time Markov chain. *Operat. Res.* **21,** 604–609.

[29] Yechiali, U. and Naor, P. (1971) Queuing problems with heterogeneous arrivals and services. *Operat. Res.* **19,** 722–734.