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SOME LIMIT THEOREMS FOR THE GENERAL SEMI-MARKOV STORAGE MODEL

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Abstract

In this paper we treat the general version of the semi-Markov storage model, introduced first by Senturia and Puri: transitions in the state of the system occur at a discrete sequence of time points, described by a two-state semi-Markov process. An input occurs at an instant of transition to state 1 and a demand for release occurs at an instant of transition to state 2.

Assuming general distributions for all the variables involved, we show that the dam contents just after the nth input converges properly in distribution as $n \to \infty$ under conditions of stability; likewise that after the nth demand. We also show that the demand lost due to shortage of stock, accumulated over instants of demand as well as over time, obeys a strong law and a central limit theorem.

STORAGE SYSTEM; CONTENTS AFTER INPUT; DEMAND; DEMAND LOST; MARKOV RENEWAL THEORY; LIMITS IN DISTRIBUTION

1. Introduction

Senturia and Puri [11], [12] introduce and study the following storage model: inputs and demands occur at discrete time points $\{T_n, n \geqq 0\}$, only one of the two being possible at any instant. $J_n = 1$ or 2 according as an input or a demand occurs at time T_n and the sequence $\{T_n, J_n, n \geqq 0\}$ is a Markov renewal process with state space $[0, \infty) \times \{1, 2\}$. Let $Z(t)$ be the contents of the dam at time t. We use the constructive definition in [12]: if n is such that $T_n \leqq t < T_{n+1}$,

$$(1.1) \qquad Z(t) = \begin{cases} Z(T_n -) + I_n & \text{if} \quad J_n = 1 \\ \max(0, Z(T_n -) - D_n) & \text{if} \quad J_n = 2 \end{cases}$$

(we have changed the notation to make it more suggestive); I_n, of course, is the input, if any, at time T_n and D_n the demand, if any. Let

$$H_{ij}(t) = P\{T_{n+1} - T_n \leqq t, J_{n+1} = j \mid J_n = i\}, \qquad i, j = 1, 2, \qquad t \geqq 0,$$

be the semi-Markov kernel of $\{T_n, J_n\}$; let $p_{ij} = H_{ij}(+\infty)$ (we do not assume that $H_{ij}(t)$ can be written as $p_{ij}H_i(t)$, as do Senturia and Puri; to begin with, we

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assume only that $H_{ij}(0)=0$). For notational convenience, we write $p_{12}=p$, $p_{21}=q$, $0<p,q<1$. $\{I_n\}$ and $\{D_n\}$ are independent processes of independent and identically distributed (i.i.d.) non-negative random variables; their means are β and α, respectively, both finite.

Senturia and Puri tackle the problem under two distinct sets of assumptions (apart from the one mentioned above), both involving exponentiality of some of the distributions involved and study the distribution of $Z(t)$, [11], [12] and of the time to first emptiness [12]; also, without assuming any special form for the distributions, they obtain limit theorems for $\{Z(t)\}$ [11].

We wish not to make any distribution assumptions and hence proceed with a 'sample function' analysis. Part of the outcome is a number of limiting results concerning the demand 'lost' due to non-availability of stock. This quantity is of importance and appears to have been largely neglected in probabilistic storage theory which concentrates mostly on contents and the time to first emptiness. Most models, of course, assume linear output at unit rate and then demand lost is precisely the dry time; and when inputs form an i.i.d. family arriving in a renewal process, the identification with the $GI/G/1$ queue provides interesting limiting results concerning the accumulated dry time (for instance, Hooke [7] shows that the accumulated dry time is asymptotically normally distributed); but when the input process is more complex (e.g. when inputs form a Markov chain, Lloyd and Odoom [8], [9]) such a convenient identification with queuing models may not be possible.

The technique used in this paper appears to have rather wide applicability to general queuing and storage models; in the last section, we shall apply it to a generalisation of the model of Lloyd and Odoom referred to above.

2. Some results from Markov renewal theory

We shall need some results from Markov renewal theory and shall provide them in this section

Suppose $S_0=0$, $S_n=\sum_{j=1}^n X_j$, $n\geqq 1$, $\{S_n,J_n,n\geqq 0\}$ a Markov renewal process taking values in $(-\infty,+\infty)\times N$, where N is the set of natural numbers or a subset thereof. Suppose that it is irreducible, ergodic (meaning that $\{J_n\}$ is irreducible and ergodic, cf. Çinlar [5]) and denote by $\{\pi_i, i\in N\}$ the stationary measure of $\{J_n\}$. Let $E_\pi(X)=\sum_{i\in N}E(X_1\mid J_0=i)\,\pi_i$, the stationary average of X; for $i\in N$ fixed, let $\{t_n(i),n\geqq 1\}$ be the 'times' of successive entry of $\{J_n\}$ to state i, i.e., $t_1(i)=\inf\{m>0\colon J_m=i\}$ and, for $n>1$, $t_n(i)=t_{n-1}(i)+\inf\{m>0\colon J_{t_{n-1}(i)+m}=i\}$; define $t_0(i)\equiv 0$ and let $Y_n(i)=\sum_{j=t_{n-1}(i)+1}^{t_n(i)}X_j$, $n\geqq 1$. We shall make repeated use of the following facts:

(2.1) (i) For $n\geqq 0$, the post-$t_n(i)$ process $\{S_{t_n(i)+m}-S_{t_n(i)}, J_{t_n(i)+m}, m\geqq 0\}$ is independent of the pre-$t_n(i)$ process $\{S_m,J_m,m\leqq t_n(i)\}$; and for $n\geqq 1$, the post-$t_n(i)$

process has the same law as $\{S_m, J_m, m \geqq 0, J_0 = i\}$; in particular, $\{Y_n(i), n \geqq 1\}$ is a family of independent variates, distributed identically for $n \geqq 2$.

(ii) For $n \geqq 2$, $E\{Y_n(i)\} = \pi_i^{-1}E_\pi(X)$ if $E_\pi(|X|) < \infty$.

(iii) Let $B^2(i) = E\{[Y_1(i) - t_1(i)E_\pi(X)]^2 \,|\, J_0 = i\}$. It is finite for one i iff it is finite for all $i \in N$; and $B^2(i)\pi_i$ is independent of i.

We also have the following result.

Theorem 1. If $E_\pi(|X|) < \infty$,

$$\text{(2.2)} \quad \text{(a)} \qquad n^{-1}S_n = n^{-1}\sum_{j=1}^{n} X_j \to E_\pi(X) \qquad \text{w.p.1}$$

and if, in addition, $B(i) < \infty$,

$$\text{(2.3)} \quad \text{(b)} \quad P\{S_n - nE_\pi(X) \leqq uB(i)\pi_i^{1/2}n^{1/2}\} \to \Phi(u), \qquad -\infty < u < \infty$$

where Φ is the standard normal distribution function.

(i) is just the strong Markov property applied to Markov renewal sequences; and we shall not prove the rest as that can be done by standard techniques (see e.g. the solidarity theorems and ergodic theorems for Markov chains, Chung [4]).

From (2.2), we may also deduce the following: (see e.g. [3])

$$\text{(2.4)} \qquad n^{-1}\min_{0 \leqq j \leqq n} S_j \to \min(0, E_\pi(X)) \qquad \text{w.p.1.}$$

3. The semi-Markov storage model

As in Section 2, let $t_n(i)$, $i = 1, 2$, be the 'time' of nth entry of $\{J_m\}$ to state i. We bring over also the rest of the notation introduced there. Write $Z(T_n -) = Z_n$, $n \geqq 1$ (so that $Z_1 = Z(0)$), $Z_{t_n(i)+1} = Z_n(i)$, $n \geqq 1$, $i = 1, 2$. In descriptive terms, $Z_n(1)$ is the contents just after the nth input, and $Z_n(2)$ that after the nth release. We first show that these two quantities have limiting distributions under a certain 'stability' condition.

From (1.1), it follows that

$$\text{(3.1)} \qquad Z_{n+1} = \max(0, Z_n + I_nC_n(1) - D_nC_n(2)) \qquad \text{(w.p.1)}$$

where $C_n(i)$ is the indicator of the event that $J_n = i$.

Let $X_n = I_nC_n(1) - D_nC_n(2)$, $n \geqq 1$; then $\{S_n, J_n, n \geqq 0\}$ is a Markov renewal process, with $\pi_1 = q(p+q)^{-1}$, $\pi_2 = p(p+q)^{-1}$; $E_\pi(X) = (q\beta - p\alpha)(p+q)^{-1}$ (the condition $E_\pi(|X|) < \infty$ is satisfied since $\beta, \alpha < \infty$), $E(Y_n(1)) = (q\beta - p\alpha)q^{-1}$, $E(Y_n(2)) = (q\beta - p\alpha)p^{-1}$, $n \geqq 2$. Let $\alpha_0(i) = 0$, $\alpha_n(i) = \sum_{j=1}^{n} Y_j(i)$, $n \geqq 1$, $i = 1, 2$. Then $\{\alpha_n(i), n \geqq 0\}$ is a (delayed) random walk. Finally, let $I'_n = I_{t_n(1)}$, $D'_n = D_{t_n(2)}$, the amounts of the nth input and demand, respectively. We use $\xrightarrow{d}$ to denote convergence in distribution and $\sim$ to denote identity in distribution.

Theorem 2. The processes $\{Z_n(i), n \geqq 1\}$, $i = 1, 2$ converge properly in distribution iff $q\beta - p\alpha < 0$ and if so

$$Z_n(1) \xrightarrow{d} \sup_{n \geqq 0} (\alpha_n(1) + I'_{n+2}) \tag{3.2}$$

and

$$Z_n(2) \xrightarrow{d} \sup_{n \geqq 0} (\alpha_n(2)) \tag{3.3}$$

for all initial contents $Z(0)$.

Proof. We prove only the first assertion. From (3.1),

$$Z_{n+1} = \max(0, Z_n + X_n) = (Z_n + X_n)^+. \tag{3.4}$$

Taking $n = t_n(1)$, and using the fact that the R's are non-negative and working backwards from (3.4), we get (we omit (1) in writing $t_n(1)$ etc.):

$$Z_n(1) = \max(0, S_{t_n} - S_{t_n - 1}, S_{t_n} - S_{t_{n-1}-1}, \cdots, S_{t_n} - S_{t_2 - 1}, S_{t_n} - S_{t_1} + Z_1(1)) \tag{3.5}$$

$$\begin{aligned} &= \max(I'_n, \alpha_n - \alpha_{n-1} + I'_{n-1}, \cdots, \alpha_n - \alpha_2 + I'_2, \alpha_n - \alpha_1 + Z_1(1)) \\ &\geqq \max(0, \alpha_n - \alpha_{n-1}, \cdots, \alpha_n - \alpha_1) \sim \max(0, \alpha_1, \cdots, \alpha_{n-1}). \end{aligned} \tag{3.6}$$

If $q\beta - p\alpha \geqq 0$, $E(Y_n(1)) \geqq 0$ and hence $\max(0, \alpha_1, \cdots, \alpha_{n-1}) \to \sup_{n \geqq 0} \alpha_n = +\infty$ w.p.1 so that $Z_n(1) \xrightarrow{d} +\infty$.

Now suppose $q\beta - p\alpha < 0$. Going back to (3.6) we observe that the families $\{R_k, t_{r-1} < k \leqq t_r, I_{t_r}\}$ $1 < r \leqq n$ are independent and have the same law and are hence 'exchangeable'. Taking then in the reverse order,

$$Z_n(1) \sim \max(I'_2, \alpha_1 + I'_3, \cdots, \alpha_{n-2} + I'_n, \alpha_{n-1} + Z_1(1)). \tag{3.7}$$

Since $E(Y_n(1)) < 0$, $\alpha_{n-1} + Z_1(1) \to -\infty$ as $n \to \infty$, and hence from (3.7),

$$Z_n(1) \xrightarrow{d} \sup_{n \geqq 0} (\alpha_n + I'_{n+2});$$

since, further, $n^{-1} I'_n \to 0$, $n^{-1}(\alpha_n + I'_{n+2}) \to E(Y_2(1)) < 0$, so that the supremum above is finite; hence the convergence is proper.

We shall now turn to a study of the demand 'lost' due to shortage of stock on hand. Going back to (3.4), consider $(Z_n + X_n)^-$ (the negative part). It is clearly zero unless $n = t_m(2)$ for some m (necessarily less than n) and $L_n = (Z_{t_n(2)} + X_{t_n(2)})^- > 0$ iff part of the nth demand is lost and then L_n equals the demand lost. We may therefore write:

$$\begin{aligned} Z_n(2) &= Z_{t_n(2)} + X_{t_n(2)} + L_n = Z_{t_n(2)-1} + X_{t_n(2)-1} + X_{t_n(2)} + L_n \\ &= \cdots = Z_{n-1}(2) + \sum^{t_n(2)} X_j + L_n = Z_{n-1}(2) + Y_n(2) + L_n. \end{aligned}$$

Hence

(3.8) $$Z_n(2) = Z(0) + \alpha_n(2) + \sum_{j=1}^{n} L_j.$$

$\sum_{j=1}^{n} L_j$ is the total amount of the first n demands that is lost due to shortage of stock. Let $\beta' = \mathrm{Var}(I)$, $\alpha' = \mathrm{Var}(D)$.

Theorem 3.

(3.9) (a) $$n^{-1} \sum_{j=1}^{n} L_j \to \max(0, p^{-1}(p\alpha - q\beta)) \qquad \text{w.p.1}$$

and

(b) if $q\beta - p\alpha < 0, \alpha', \beta' < \infty$,

$$P\left\{\sum_{j=1}^{n} L_j - np^{-1}(p\alpha - q\beta) \leqq un^{1/2}\sigma\right\} \to \Phi(u), \qquad -\infty < u < \infty$$

where $\sigma^2 = qp^{-1}\beta' + qp^{-2}\beta^2(2 - p - q) + \alpha'$.

Proof. From (3.4) it follows that

(3.10) $$Z_{n+1} = S_n - \min_{1 \leqq j < n} (S_j, - Z(0))$$

and hence, by (2.2) and (2.4),

$$n^{-1} Z_n \to \max(0, (p+q)^{-1}(q\beta - p\alpha)) \qquad \text{w.p.1.}$$

Using this in (3.8), we get (3.9).

The central limit result also follows easily from (3.8) since $\{\alpha_n(2)\}$ is a random walk to which the classical central limit theorem applies, and $n^{-1/2} Z_n(2) \xrightarrow{P} 0$ (stochastic convergence) when $q\beta - p\alpha < 0$, as a consequence of (3.4). It remains to show that $\sigma^2 = \mathrm{Var}(Y_n(2))$, and that is only tedious.

Until now the distributions H_{ij} have not explicitly come into the picture since we have been working in the discrete set of transition epochs. We shall now move to continuous time; let $L(t)$ be the total amount of demand lost during $[0, t]$. Let

$$M_i(t) = \sup\{n \geqq 0: T_{t_n(i)} \leqq t\}, \quad i = 1, 2,$$

$$M(t) = \sup\{n \geqq 0: T_n \leqq t\}, \quad t \geqq 0.$$

Then

(3.11) $$L(t) = \sum_{j=1}^{M_2(t)} L_j.$$

Let

$$\theta_{ij}^{(k)} = \int_{t=0}^{\infty} t^k dH_{ij}(t) \leqq \infty, \qquad i, j = 1, 2, k \geqq 1;$$

$$\theta_i^{(k)} = \theta_{i1}^{(k)} + \theta_{i2}^{(k)}.$$

For notational convenience we omit the superscript when $k = 1$. We assume hereafter that $\theta_i < \infty$.

Theorem 4.

$$\text{(3.12)} \quad \text{(a)} \qquad t^{-1}L(t) \to \max(0, (q\theta_1 + p\theta_2)^{-1}(p\alpha - q\beta)) \qquad \text{w.p.1}$$

and

(b) if $q\beta - p\alpha < 0, \alpha', \beta', \theta_1^{(2)}, \theta_2^{(2)} < \infty$,

$$P\{L(t) - t(q\theta_1 + p\theta_2)^{-1}(p\alpha - q\beta) \leqq ut^{1/2}q^{1/2}(q\theta_1 + p\theta_2)^{-1/2}\sigma'\} \to \Phi(u),$$

where σ' is given by

$$(\sigma')^2 = pq^{-1}\alpha' + \alpha^2pq^{-2}(2 - q) + \beta' + \beta^2 - 2\beta p\alpha q^{-1} - 2\beta\delta q^{-1}(q\theta_1 + p\theta_2)$$

$$\text{(3.13)} \qquad + 2\delta\alpha q^{-2}(q\theta_{12} + p\theta_{21} + 2p\theta_{22}) + \delta^2 q^{-2}[q^2\theta_1^{(2)} + pq\theta_2^{(2)} + 2\theta_2(q\theta_{12} + p\theta_{22})]$$

where $\delta = (q\theta_1 + p\theta_2)^{-1}(q\beta - p\alpha)$.

Proof.

(a) Since $t^{-1}M(t) \to (q\theta_1 + p\theta_2)^{-1}(p + q)$ (Çinlar [5]), (3.11) and (3.9) are easily seen to imply (3.12).

(b) To prove Theorem 4(b) we need some auxiliary results which are of interest in themselves. The proofs can be simplified using the independence of the basic processes but we shall not do so, with a view to facilitating generalisations (see concluding remarks). A wet period is defined as an interval of time during all of which the system is non-empty.

Lemma 1. The total number of wet periods is finite iff $q\beta - p\alpha > 0$.

Proof. The total number of wet periods is finite iff $Z_n > 0$ for all sufficiently large n; i.e., (by (3.10)), iff $S_n > \min_{1 \leqq j < n}(S_j, -Z(0))$ for all large n, which implies that $\{S_n\}$ has an a.s. finite minimum; if $q\beta - p\alpha \leqq 0$, $E(Y_n(2)) \leqq 0$ and hence $\liminf_{n\to\infty} S_n \leqq \liminf_{n\to\infty} \alpha_n(2) = -\infty$, a contradiction. If $q\beta - p\alpha > 0$, (2.2) implies that $S_n \to +\infty$ w.p.1, so that $S_n > -Z(0)$ for large enough n, which implies that $Z_n > 0$ for all large enough n.

We note an important fact: since $\{S_n\}$ can decrease only at epochs of release, its successive minima coincide with those of $\{\alpha_n(2)\}$.

Suppose $\{W_n, n \geqq 1\}$ are the transition epochs that mark the end of dry periods; if $Z(0) = 0$, W_1 is the epoch of the first input, i.e., $W_1 = t_1(1)$. In any case, W_n is a stopping time for $\{S_n, J_n\}$ and $\{W_n - W_1, J_{W_n}, n \geqq 1\}$ is a Markov renewal process; and since $J_{W_n} \equiv 1$, $\{W_n, n \geqq 1\}$ is a (delayed) renewal process. We shall denote the increments to it by W.

A wet cycle is a wet period followed by a dry period. If $Z(0) = 0$, the initial dry period will be called the first wet cycle. Let V_n be the time at which the nth wet cycle ends. Then, as with $\{W_n\}$, $\{V_n, n \geqq 1\}$ is a (delayed) renewal process. Increments to it will be denoted by V (V_n refers to continuous time, whereas W_n refers to the embedded discrete transition epochs; in fact, $V_n = T_{W_n}$).

Lemma 2. If $q\beta - p\alpha \leqq 0$, $E(W)$ and $E(V)$ are finite iff $q\beta - p\alpha < 0$.

Proof. Let W^* denote a random variable distributed as the number of the transition epoch at which a wet period starting with contents zero (and J necessarily in state 1) ends. Then, $W^* \sim \inf\{n\colon Z_{n+1} = 0\}$, is finite by Lemma 1. Let $t_0(2) = 0$, $h(n) = \sup\{m \geqq 0\colon t_m(2) \leqq n\}$.
Then, from (3.10),

$$\begin{aligned} P(W^* > n) &= P(S_1 > 0, S_2 > 0, \cdots, S_n > 0) \\ &= P(h(n) = 0) + P(\alpha_1 > 0, \cdots, \alpha_{h(n)} > 0;\ h(n) > 0) \qquad (\alpha_j = \alpha_j(2)) \\ &= P(h(n) = 0) + \sum_{r=1}^{n} P(\alpha_1 > 0, \cdots, \alpha_r > 0, t_r(2) \leqq n < t_{r+1}(2)). \end{aligned}$$

Summing over n, and interchanging the order of the second summation,

$$\begin{aligned} E(W^*) &= \sum_{n=0}^{\infty} P(h(n) = 0) + \sum_{r=1}^{\infty}\sum_{n=r}^{\infty} P(\alpha_1 > 0, \cdots, \alpha_r > 0, t_r(2) \leqq n < t_{r+1}(2)) \\ &= \sum_{n=0}^{\infty} P(h(n) = 0) + \sum_{r=1}^{\infty} P(\alpha_1 > 0, \cdots, \alpha_r > 0) E(t_{r+1}(2) - t_r(2)) \end{aligned}$$

by independence of the pre- and post-$t_r(2)$ processes, (see (2.1) (i)). Hence

$$E(W^*) = \sum_{n=0}^{\infty} P(h(n) = 0) + (m - 1)p^{-1}(p + q) \tag{3.14}$$

where m is the average of the first (weak) descending ladder epoch for the random walk $\{\alpha_n(2)\}$; the first sum in (3.14) is the mean number of steps in the first passage of $\{J_n\}$ from state 1 to state 2 and is finite; and $m < \infty$ iff $E(Y_n(2)) < 0$ (Feller [6]), i.e., $q\beta - p\alpha < 0$. Also, the mean of $W - W^*$ is in any case finite since it is the number of steps in the first passage of $\{J_n\}$ from state 2 to state 1. The first assertion of the lemma is proved; the second may be proved likewise.

Proof of Theorem 4(*b*). We have the conservation relation:

$$Z(t)=\sum_{j=1}^{M(t)} I_jC_j(1)-\left[\sum_{j=1}^{M(t)} D_jC_j(2)-L(t)\right]$$
$$(3.15) \qquad = S_{M(t)}+L(t).$$

Now, $Z(t)$ is not larger than the total input during the wet cycle running at time t; and it is an easy matter, using Smith's [13] renewal theorem, to show that this latter quantity has a proper limiting distribution as $t\to\infty$ if $E(V)<\infty$, i.e., (by Lemma 2) if $q\beta-p\alpha<0$. Hence $t^{-1/2}Z(t)\xrightarrow{P}0$ (for the special cases treated by Senturia and Puri this follows from the fact that $Z(t)$ has a proper limiting distribution as $t\to\infty$).

Now, by definition of $M_1(t)$, $t_{M_1(t)}(1)\leqq M(t)$, and $S_{M(t)}=S_{M(t)}-\alpha_{M_1(t)}+\alpha_{M_1(t)}$ (we omit (1) from $\alpha_n(1)$, $t_n(1)$, etc).

Consider

$$|S_{M(t)}-\alpha_{M_1(t)}|\leqq \sum_{j=t_{M_1(t)}+1}^{t_{M_1(t)+1}} |X_j|;$$

again, using the strong Markov property, (2.1) (i), and the renewal theorem we may show that this quantity has a proper limiting distribution as $t\to\infty$ since $E(T_{t_1(1)}\mid J_0=1)<\infty$. Hence $t^{-1/2}(S_{M(t)}-\alpha_{M_1(t)})\xrightarrow{P}0$. It therefore suffices, by (3.15), to show that Theorem 4(b) is true with $L(t)$ replaced by $-\alpha_{M_1(t)}$. Now

$$\alpha_{M_1(t)}-t(q\theta_1+p\theta_2)^{-1}(q\beta-p\alpha)$$
$$(3.16) \qquad =\sum_{j=1}^{M_1(t)}[Y_j-(T_{t_j}-T_{t_{j-1}})(q\theta_1+p\theta_2)^{-1}(q\beta-p\alpha)]$$
$$+(T_{t_{M_1(t)}}-t)(q\theta_1+p\theta_2)^{-1}(q\beta-p\alpha).$$

Denote the summand in the first term of (3.16) by K_j. Since

$$E(T_{t_j}-T_{t_{j-1}})=q^{-1}(p+q)(p+q)^{-1}(q\theta_1+p\theta_2)=q^{-1}(q\theta_1+p\theta_2),$$

(see (2.1)(ii)), $E(K_j)=0$. Its variance can be seen to be $(\sigma')^2$. As a consequence of the renewal theorem,

$$t^{-1/2}(T_{t_{M_1(t)}}-t)\xrightarrow{P}0,$$

and hence, by (3.16), it is sufficient to concentrate on $\Sigma_1^{M_1(t)}K_j$.

Let t^* be the integer part of $(q\theta_1+p\theta_2)^{-1}qt$. Since $t^{-1}M_1(t)\to q(q\theta_1+p\theta_2)^{-1}$, we may use the same technique as Chung ([4], p. 100) to show that

$$t^{-1/2}\left(\sum_1^{M_1(t)}K_j-\sum_1^{t^*}K_j\right)\xrightarrow{P}0.$$

Since

$$P\left\{\sum_{1}^{t^*} K_j \leqq u\sigma' t^{1/2} q^{1/2}(q\theta_1 + p\theta_2)^{-1/2}\right\} \to \Phi(u) \quad \text{as } t \to \infty$$

by the classical central limit theorem for the i.i.d. sequence $\{K_j\}$, the proof is complete.

Remarks. The model may be generalised to allow dependence between the sojourn times $\{T_n - T_{n-1}\}$, inputs $\{I_n\}$ and demands $\{D_n\}$. For instance we may assume that $\{T_n - T_{n-1}, I_n, D_n, J_n\}$ forms a Markov renewal sequence. A glance at the proofs of our results shows that they are worded sufficiently generally to be valid verbatim in this general case too; the only casualties are the explicit evaluations for $E_\pi(X)$ and the norming constants σ, σ' in the central limit results.

4. A storage model in a Markovian environment

Lloyd and Odoom [8], [9] and Ali Khan and Gani [1] have made a detailed study of the following storage model: the model is in discrete time; inputs $\{I_n, n \geqq 1\}$ form a finite Markov chain with state space N, a subset of the natural numbers. Demand is unit and is supplied if the contents are not zero. Generalising the model we assume that demands $\{D_n, n \geqq 1\}$ form an i.i.d. family of positive integer-valued random variables independent of the input Markov chain (Pakes [10] takes $D_n \equiv M$, a constant), and obtain a limit theorem for the (accumulated) demand lost.

Let Z_n be the contents just before the nth input and release. Then

$$Z_{n+1} = (Z_n + I_n - D_n)^+ = (Z_n + X_n)^+, \qquad X_n = I_n - D_n, \quad n \geqq 1. \tag{4.1}$$

Let $S_0 = 0$, $S_n = \Sigma_{j=1}^n X_j$, $n \geqq 1$; then $\{S_n, I_n, n \geqq 0\}$ is a Markov renewal process (I_0 is the initial state for the input Markov chain), and $\{Z_{n+1}, I_n, n \geqq 0\}$ is a Markov chain. We assume that it is irreducible. The stationary distribution of $\{I_n\}$ is denoted by $\{\pi_i, i \in N\}$.

Let $L_n = (Z_n + X_n)^-$. It is the demand lost at 'time' n. We wish to prove results analogous to Theorem 4 for $\{L_n\}$. We require a preliminary lemma. Let $E(D) = d < \infty$.

Lemma 3. $\{Z_n\}$ is a stochastically bounded family if $\Sigma_{j \in N} j\pi_j < d$.

Proof. Suppose first that $D_n \leqq D$ for all n, D a constant. Then the irreducible Markov chain $\{Z_{n+1}, I_n\}$ is ergodic, by Theorem 5 of Balagopal [2], and hence $\{Z_n\}$ is stochastically bounded. For the general case, let $D_n' = \min(D, D_n)$. Then $D_n' \to D_n$ monotonically as $D \to \infty$ so that $E(D_n')$ also increases to $E(D_n)$, by the monotone convergence theorem. Choose D so that $\Sigma j\pi_j < E(D_n')$. Define Z_n'

using D'_n in place of D_n. Then $\{Z'_n\}$ is stochastically bounded, by what precedes and since, for each n, Z'_n clearly dominates Z_n stochastically, $\{Z_n\}$ is also stochastically bounded.

Theorem 5.

$$\text{(4.2)}\quad \text{(a)}\qquad n^{-1}\sum_{j=1}^{n} L_j \to \max\Big(0, d - \sum_{j\in N} j\pi_j\Big) \qquad \text{w.p.1}$$

and

(b) if $\sum_{j\in N} j\pi_j < d, E(D_n^2) < \infty,$

$$P\Big\{\sum_{j=1}^{n} L_j - n\Big(d - \sum_{j\in N} j\pi_j\Big) \leqq un^{1/2}B\Big\} \to \Phi(u), \qquad -\infty < u < \infty$$

where

$$\text{(4.3)}\qquad B^2 = B^2(i)\pi_i, \quad B^2(i) \text{ as defined in (2.1) (iii).}$$

Proof.

(a) From (4.1) we have

$$Z_{n+1} = Z_n + L_n + X_n = Z_1 + \sum_{j=1}^{n} L_j + S_n.$$

(4.4) It is also easy to show that

$$Z_{n+1} = S_n - \min_{1\leqq j<n} (S_j, - Z_1).$$

Now, $E_\pi(X) = \Sigma_{j\in N} j\pi_j - d$ and hence, by (2.2) and (2.4),

$$n^{-1}Z_n \to \max\Big(0, \sum_{j\in N} j\pi_j - d\Big) \quad \text{w.p.1.}$$

Using this and (2.2) in (4.4), we get (4.2).

(b) By Lemma 3, $n^{-1/2}Z_n \xrightarrow{P} 0$ and hence (4.3) follows from (4.4) by (2.3). We need only note that $B < \infty$ iff $E(D_n^2) < \infty$.

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